# Gell-Mann - Oakes - Renner relation in a magnetic field at finite temperature.

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**Abstract**

In the first order of chiral perturbation theory the corrections to $F_{\pi^0}$ and $M_{\pi^0}$ in a magnetic field at finite temperature have been found. It was shown that they are shifted in such a manner that Gell-Mann - Oakes - Renner relation remains valid under these conditions.

PACS:11.10.Wx,12.38.Aw,12.38.Mh

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1. The investigation of the vacuum properties and hadron structure behavior under influence of the various external factors is known to be one of the interesting problems in quantum field theory. At low energies strong interactions can be adequately described by chiral dynamics of light pseudoscalar particles - $\pi$-mesons, i.e. chiral effective theory [1,2]. Besides, in the studying of QCD hadron phase the low-energy relations at finite temperature [3] in a magnetic field [4] are very useful. One of the important phenomenon of low energy physics of pions is Gell-Mann - Oakes - Renner (GOR) relation [5], establishing the connection between pion mass $M_\pi$, axial coupling constant $F_\pi$ and quark condensate $\langle \bar{q}q \rangle$. Validity of GOR relation at finite temperature was investigated in [6]. It was found that GOR relation is not affected by low temperature. Similar consideration of GOR relation in external magnetic field was done in [7]. In the presence of the field, the axial $SU_A(2)$ symmetry is broken down to $U^3_A(1)$ corresponding to chiral rotation of $u-$ and $d-$ quarks with opposite phases (the singlet axial symmetry is broken already in the absence of the field due to the anomaly ). The formation of the condensate breaks down this remnant $U^3_A(1)$ symmetry spontaneously leading to appearance of a Goldstone boson, the $\pi^0$-meson. Meanwhile, charged pions acquire a gap in the spectrum $\propto \sqrt{eH}$ and are not goldstones anymore.

Phase structure of QCD vacuum in a constant magnetic field $H$ at low temperature $T$ was explored in [8]. In the framework of chiral perturbation theory (ChPT) the dependence of the quark condensate upon $T$ and $H$ was calculated and showed that shift of the condensate is not a simple sum of temperature ($\sim T^2/F_\pi^2$) and magnetic ($\sim H/F_\pi^2$) corrections. Orbital diamagnetism of charged pion gas results in the additional, "cross" term. In the present paper we study GOR relation in a magnetic field at finite temperature.

2. At low temperatures $T \ll T_c$ ($T_c$ is the temperature of the phase transition with chiral symmetry restoring) and in the region of the weak magnetic fields [1] $H \ll \mu^2_{hadr} \sim (4\pi F_\pi)^2$ (electric charge $e$ is absorbed in $H$) characteristic momenta in loops are small and QCD is effectively described by chiral lagrangian $L_{eff}$ [2] which can be represented as an expansion in powers of the external momenta (derivatives) and the quark masses

$$L_{eff} = L^{(2)} + L^{(4)} + L^{(6)} + \cdots \tag{1}$$

The leading term $L^{(2)}$ in (1) is similar to the Lagrangian of nonlinear $\sigma$-model in external field $V_\mu$

$$L^{(2)} = \frac{F_\pi^2}{4}\mathrm{Tr}(\nabla_\mu U^+ \nabla_\mu U) + \Sigma \mathrm{ReTr}(\mathcal{M}U^+), \tag{2}$$

$$\nabla_\mu U = \partial_\mu U - i\,[U, V_\mu]\,.$$

Here $U$ stands for an unitary $SU(2)$ matrix (for two flavors), $F_\pi = 93$ Mev is the pion decay constant and parameter $\Sigma$ has the meaning of the quark condensate $\Sigma = |\langle \bar{u}u \rangle| = |\langle \bar{d}d \rangle|$. The external Abelian magnetic field $H$, aligned along $z$-axis, corresponds to $V_\mu(x) = (\tau^3/2)A_\mu(x)$, where vector potential $A_\mu$ may be chosen as $A_1(x) = -Hx_2$. The difference of light quark masses $m_u - m_d$ enters in the effective Lagrangian (1) only quadratically. To calculate the quantities which we are interested in the chiral limit, we can put $m_u = m_d = 0$ from the very beginning. It means that we can take mass matrix to be diagonal $\mathcal{M} = m\hat{I}$.

In one-loop approximation of ChPT the behavior of quark condensate in the presence of magnetic field and finite temperature was found in [8]. It proved to be enough to expand $L_{eff}$ only up to quadratical terms of pion fields, i.e. neglect pion interactions. The result for the quark condensate is [8]

$$\frac{\Sigma(T,H)}{\Sigma} = 1 - \frac{T^2}{24F_\pi^2} + \frac{H}{(4\pi F_\pi)^2}\ln 2 - \frac{H}{2\pi^2 F_\pi^2}\varphi\left(\frac{\sqrt{H}}{T}\right)$$

[1] A transition to the chiral limit that $M_\pi^2 \ll H$. It was shown in [4] that the parameter of the ChPT expansion in a magnetic field is $\xi = H/(4\pi F_\pi)^2$. Thus, the domain of ChPT validity in a magnetic field is $M_\pi^2/(4\pi F_\pi)^2 \ll \xi < 1$. In the chiral limit, the axial constant $F_\pi(M_\pi \to 0) \to const \approx 80$ MeV; hence, we have $0 < \xi < 1$.

$$\varphi(\lambda) = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{dx}{\omega_n(x)(\exp(\lambda\omega_n(x)) - 1)}, \quad \omega_n(x) = \sqrt{x^2 + 2n + 1} \tag{3}$$

In the chiral limit the function $\varphi$ depends only on the ratio $\lambda = \sqrt{H}/T$ and has the following asymptotic behavior [8]

$$\varphi(\lambda \gg 1) = \sqrt{\frac{\pi}{2\lambda}} e^{-\lambda} + O(e^{-\sqrt{3}\lambda}), \tag{4}$$

$$\varphi(\lambda \ll 1) = \frac{\pi^2}{6\lambda^2} + \frac{7\pi}{24\lambda} + \frac{1}{4}\ln\lambda + C + \frac{\zeta(3)}{48\pi^2}\lambda^2 + O(\lambda^4). \tag{5}$$

Here $C = 1/4(\gamma - \ln 4\pi - 1/6)$, $\gamma = 0.577...$ - Euler constant and $\zeta(3) = 1.202...$ is Riemann zeta-function.

To check Gell-Mann - Oakes - Renner relation for $\pi^0$ meson we need to know the expressions for $M^2_{\pi^0}(H,T)$ and $F^2_{\pi^0}(H,T)$. Firstly, let us find the shift of $\pi^0$ mass in magnetic field at finite temperature in the leading order of ChPT. In this order it is enough to expand $L^{(2)}$ up to 4-pion vertices. Choosing Weinberg parameterization for matrix $U$

$$U = \sigma + \frac{i\pi^a\tau^a}{F_\pi}, \quad \sigma^2 + \frac{\vec{\pi}^2}{F_\pi^2} = 1, \tag{6}$$

we get the next order terms of lagrangian $L^{(2)}$

$$\Delta L^{(2)} = \frac{1}{2F_\pi^2}[\pi^0\partial_\mu\pi^0 + \partial_\mu(\pi^+\pi^-)]^2 - \frac{M_\pi^2}{8F_\pi^2}[2\pi^+\pi^- + (\pi^0)^2] \tag{7}$$

Diagrams, contributing to shift of $M^2_{\pi^0}(H,T)$ are represented in Fig. 1.


Figure 1:

Using vertices from (7), $M^2_{\pi^0}(H,T)$ can be represented as

$$\frac{M^2_{\pi^0}(H,T)}{M^2_{\pi^0}} = 1 - \frac{1}{2F^2_{\pi^0}} D^R_T(0) + \frac{1}{F^2_{\pi^0}} D^R_{T,H}(0), \tag{8}$$

where $D^R_T(0)$ and $D^R_{T,H}(0)$ are propagators of scalar particles at coinciding initial and final points. Index "$R$" means that we have subtracted from propagator its value at $H = 0$, $T = 0$. The second term in (8) corresponds to Fig.1 when the particles running in the loop are $\pi^0$-mesons noninteracting with magnetic field. Temperature propagator $D^R_T(0)$ can be represented in the form

$$D^R_T(0) = D_T(x,x) - D_0(x,x) = T\sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\omega_n^2 + \omega^2(\mathbf{p})} - \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + M_\pi^2}, \tag{9}$$

where $\omega_n = 2\pi n/\beta$, $\beta = 1/T$, $\omega^2(\mathbf{p}) = \mathbf{p^2} + M_\pi^2$. Summing over Matsubara frequencies [2] in (9) we obtain (in the chiral limit $M_\pi^2 \to 0$)

$$D_T^R(0) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\omega(e^{\beta\omega} - 1)} = \frac{T^2}{12} \tag{10}$$

The third term in Eq.(8) corresponds to Fig.1 with charged $\pi^\pm$-mesons in the loop and $D_{T,H}^R(0)$ denotes temperature propagator (at coinciding points) of scalar particle moving in constant magnetic field

$$D_{T,H}^R(0) = \frac{H}{2\pi} \sum_{k=0}^{\infty} T \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp_z}{2\pi} \frac{1}{\omega_n^2 + \omega_H^2(k, p_z)} - \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + M_\pi^2} \tag{11}$$

and $\omega_H^2(k, p_z) = p_z^2 + M_\pi^2 + H(2k+1)$ are Landau levels. The first term in (11) reproduces the summation over eigenvalues of Hamiltonian of charged scalar particle in magnetic field and the degeneracy multiplicity of $H/2\pi$ has been taken into account for the Landau levels. It is convenient to split $D_{T,H}^R(0)$ into two parts: "vacuum" ($T = 0$) and "matter" ($T \neq 0$) pieces

$$D_{T,H}^R(0) = D_{0,H}^R(0) + \tilde{D}_{T,H}(0), \tag{12}$$

where (in the chiral limit $M_\pi^2 \to 0$)

$$D_{0,H}^R(0) = \frac{H}{8\pi^2} \sum_{k=0}^{\infty} \int_{-\infty}^{+\infty} \frac{dp_z}{\omega_H(k, p_z)} - \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} = -\frac{H}{16\pi^2} \ln 2$$

$$\tilde{D}_{T,H}(0) = \frac{H}{4\pi^2} \sum_{k=0}^{\infty} \int_{-\infty}^{+\infty} \frac{dp_z}{\omega_H(e^{\beta\omega_H} - 1)} = \frac{H}{2\pi^2} \varphi\left(\frac{\sqrt{H}}{T}\right) \tag{13}$$

Substituting (10) and (13) in (8) we find one-loop correction to $M_{\pi^0}^2$

$$\frac{M_{\pi^0}^2(H, T)}{M_{\pi^0}^2} = 1 - \frac{T^2}{24F_\pi^2} - \frac{H}{16\pi^2} \ln 2 + \frac{H}{2\pi^2} \varphi\left(\frac{\sqrt{H}}{T}\right) \tag{14}$$

To calculate renormalization of axial coupling constant $F_{\pi^0}$ we consider the correlator of two axial currents

$$\Pi_{\mu\nu}^A = i \int d^4x e^{iqx} \langle A_\mu(x) A_\nu(0) \rangle, \tag{15}$$

where $A_\mu = \bar{q}\gamma_\mu\gamma_5(\tau^3/2)q$ is the third component of axial current. As $F_\pi$ and its shift are nonzero in the chiral limit, we put $M_\pi^2 = 0$ from the very beginning. Then this correlator becomes transverse ant takes the form

$$\Pi_{\mu\nu}^A = F_\pi^2 (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) \tag{16}$$

[2]The following formula was used for summation (see, e.g. [9])

$$\sum_{n=-\infty}^{+\infty} \frac{1}{(2\pi n/\beta)^2 + \omega^2} = \frac{\beta}{2\omega} + \frac{\beta}{\omega(e^{\beta\omega} - 1)}$$

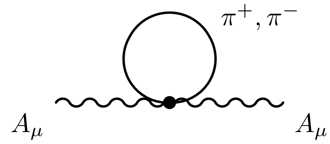


Figure 2:

and we will calculate the coefficient before $g_{\mu\nu}$. In the first order of ChPT the relevant part of chiral lagrangian describing the interaction of pions with sources of axial current $a_\mu$ is

$$\Delta L_A = \frac{1}{2} F_\pi^2 a_\mu^2 - a_\mu^2 \pi^+ \pi^- \tag{17}$$

The shift of $F_{\pi^0}$ is given by diagram in Fig. 2 and can be written in the form

$$\frac{F_{\pi^0}^2(H,T)}{F_{\pi^0}^2} = 1 - \frac{2}{F_{\pi^0}^2} D_{T,H}^R(0), \tag{18}$$

Using relations for $D_{T,H}^R$ we finally obtain correction to axial coupling constant in the magnetic field $H$ at finite temperature $T$.

$$\frac{F_{\pi^0}^2(H,T)}{F_{\pi^0}^2} = 1 + \frac{H}{8\pi^2} \ln 2 - \frac{H}{\pi^2} \varphi \left( \frac{\sqrt{H}}{T} \right) \tag{19}$$

Making use of (14), (19) and (3) we find the following equation

$$F_{\pi^0}^2(T,H) M_{\pi^0}^2(T,H) = 2m\Sigma(T,H), \tag{20}$$

which means that in the first order of ChPT Gell-Mann - Oakes - Renner relation remains valid in the magnetic field at finite temperature. Hence, the scheme of "soft" breaking of chiral symmetry by quark masses is not affected by magnetic field and temperature.

Let us consider different limiting cases. When magnetic field is switched off we reproduce well-known results at finite temperature [6]

$$\frac{\Sigma(T)}{\Sigma} = 1 - \frac{T^2}{8F_\pi^2}, \quad \frac{M_\pi^2(T)}{M_\pi^2} = 1 + \frac{T^2}{24F_\pi^2}, \quad \frac{F_\pi^2(T)}{F_\pi^2} = 1 - \frac{T^2}{6F_\pi^2}. \tag{21}$$

If we set $T = 0$ then we get the results obtained in [7]

$$\frac{\Sigma(H)}{\Sigma} = 1 + \frac{H \ln 2}{(4\pi F_\pi)^2}, \quad \frac{M_\pi^2(H)}{M_\pi^2} = 1 - \frac{H \ln 2}{(4\pi F_\pi)^2}, \quad \frac{F_\pi^2(H)}{F_\pi^2} = 1 + \frac{2H \ln 2}{(4\pi F_\pi)^2}. \tag{22}$$

In both limiting cases: $\sqrt{H}/T \ll 1$ and $\sqrt{H}/T \gg 1$ all corrections are governed by $\varphi$ and can be easily found from (4) and (5), and for the quark condensate these corrections were obtained in [8].

It has been shown in [8] that the quark condensate is "frozen" by the magnetic field when both temperature $T$ and magnetic field $H$ are increased according to the $H = const \cdot T^2$ law. The same effect can be observed for $M_{\pi^0}(H,T)$ and $F_{\pi^0}(H,T)$ behavior. To find a regime of $\pi^0$ mass "freezing" we should solve the equation followed from (14)

$$1 + \frac{3}{2\pi^2} \lambda^2 \ln 2 - \frac{12}{\pi^2} \lambda^2 \varphi(\lambda) = 0 \tag{23}$$

Numerically, $\lambda_M \approx 0.069...$. Hence, $\pi^0$ mass remains unchanged when $H = \lambda_M T^2$. Similar consideration can be done for $F_{\pi^0}(H,T)$. In this case appropriate equation is

$$\varphi(\lambda) = \frac{1}{8}\ln 2 \tag{24}$$

and $\lambda_F \approx 2.323...$

3. In the present paper we found the behavior of $F_{\pi^0}$ and $M_{\pi^0}$ constants in a magnetic field at finite temperature. It was demonstrated that for $F_{\pi^0}$ and $M_{\pi^0}$ effect of "freezing" takes place when both temperature $T$ and magnetic field $H$ increase according to the $H = const \cdot T^2$ law and two appropriate constants $\lambda_F$ and $\lambda_M$ were numerically found. It was shown that Gell-Mann - Oakes - Renner relation for $\pi^0$ meson remains valid, i.e. the mechanism of "soft" breaking of chiral symmetry by light quark masses is not affected by magnetic field and temperature.

N.A. acknowledges the financial support of RFBR grant 00-02-17836 and INTAS grant CALL 2000 No.110. I.S. acknowledges the financial support of CRDF No. RP2-2247, RFBR No. 00-02-17808, and grant INTAS 2000-587.

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